SPMR: A Family of Saddle-Point Minimum Residual Solvers

Contact Information
Ron Estrin
ICME PhD Candidate,
Stanford University
restrin@stanford.edu

Ron Estrin and Chen Greif
Institute for Computational and Mathematical Engineering
Stanford University

Contact Information
Chen Greif
Professor, University of British Columbia
greif@cs.ubc.ca

The Problem

We are interested in solving

$$\mathcal{K}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}A & G_1^T\\ G_2 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}f\\g\end{bmatrix}, \quad (1)$$

where $A \in \mathbb{R}^{n\times n}$, $G_1, G_2 \in \mathbb{R}^{m\times n}$, $f \in \mathbb{R}^n$, and $g \in \mathbb{R}^m$, with $m < n$.

We design iterative methods where we require that either:

- A is efficiently invertible
- can efficiently project to null-space of G_1, G_2

Dual Saddle-Point System

For $g = 0$, the *dual-saddle point system* related to (1) is

$$\mathcal{K}_D\begin{bmatrix}p\\q\end{bmatrix} = \begin{bmatrix}A & AH_2\\ AH_1^T & 0\end{bmatrix}\begin{bmatrix}p\\q\end{bmatrix} = \begin{bmatrix}0\\-H_1^T f\end{bmatrix}, \quad (2)$$

where $G_1H_1 = G_2H_2 = 0$ are null-space operators.

There exists a solution to (2) such that $x = H_2 q = -p$.

SPMR Family Tree

We have a family of 4 methods, depending on the properties of the problem.

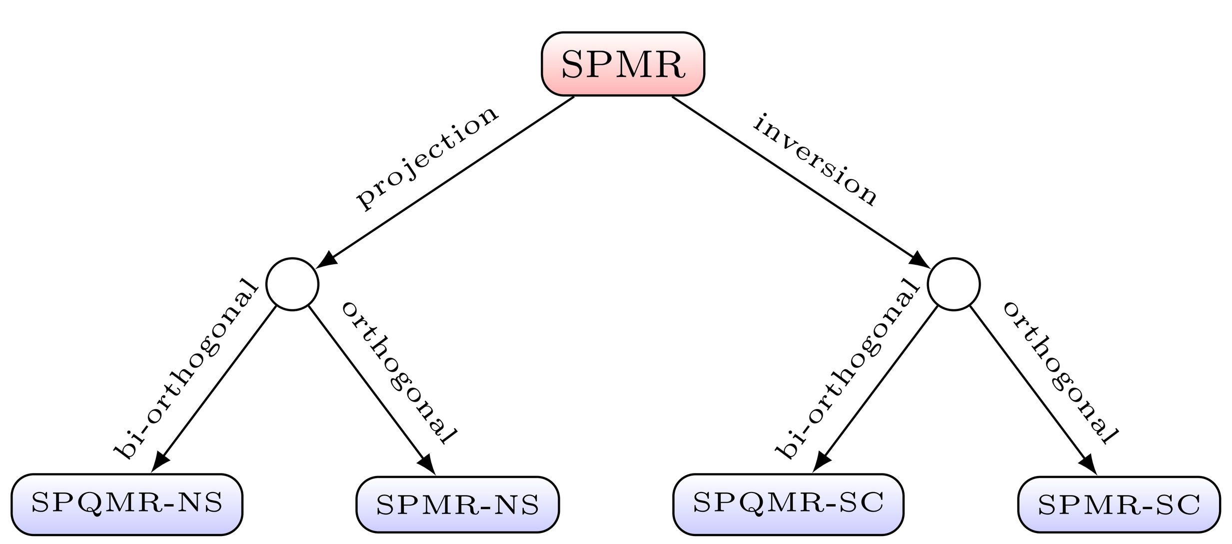


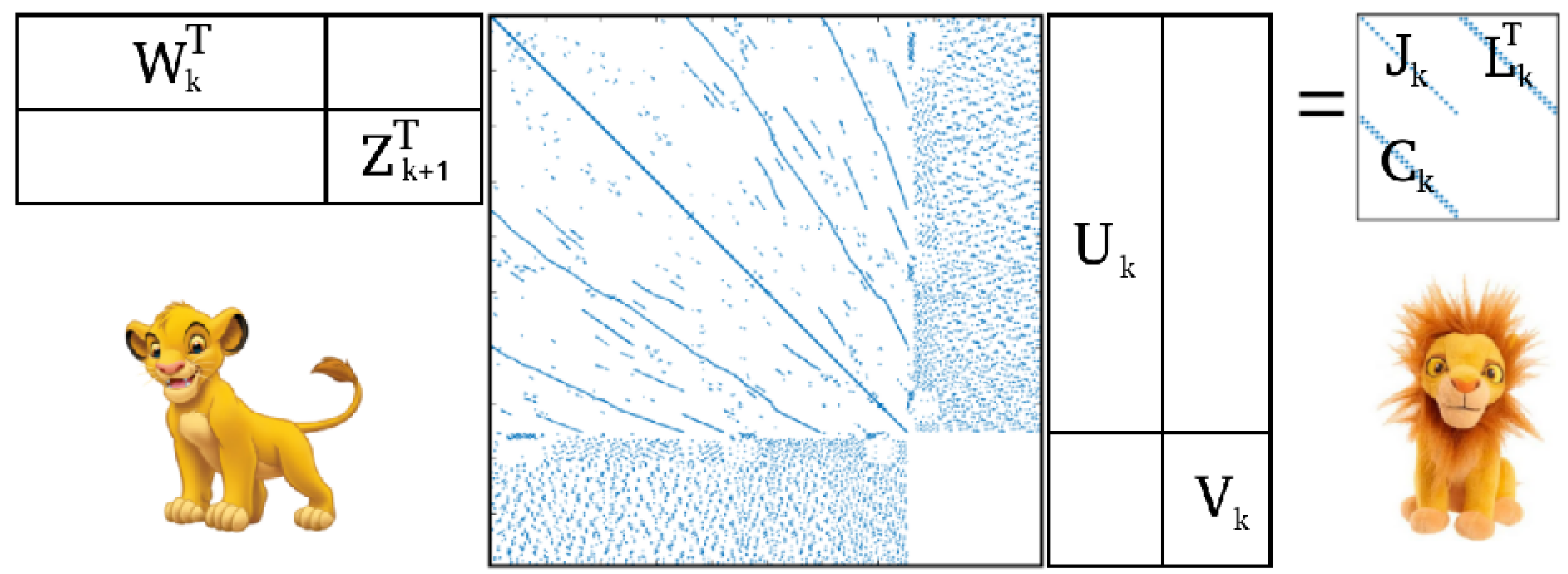
Figure: Various versions of SPMR.

A is efficiently **invertible**: **right** branch (–SC)
Efficient **projection** to $\ker(G_1)$, $\ker(G_2)$: **left** branch (–NS)

For **orthogonal** search bases: **right** sub-branch (SPMR)
For **bi-orthogonal** search bases: **left** sub-branch (SPQMR)

SIMBA and SIMBO

Lanczos-like process to construct bases U_k, V_k, W_k, Z_k to project to smaller saddle-point matrix.



SIMBA: SIMultaneous Bidiagonalization via A-conjugacy $\implies$ SPMR methods
SIMBO: SIMultaneous Bidiagonalization via bi-Orthogonality $\implies$ SPQMR methods

SIMBA

Relationships deriving process:

$$G_1^T V_k = AU_kJ_kL_k^T, \qquad W_k^T A U_k = J_k,$$
$$G_1 W_k = V_{k+1}B_k, \qquad V_k^T V_k = I,$$
$$G_2^T Z_k = A^T W_k J_k M_k^T, \qquad Z_k^T Z_k = I,$$
$$G_2 U_k = Z_{k+1}C_k,$$

SIMBO

Relationships deriving process:

$$G_1^T V_k = AU_kJ_kL_k^T, \qquad W_k^T A U_k = J_k,$$
$$G_1 W_k = Z_{k+1}B_k, \qquad Z_k^T V_k = I,$$
$$G_2^T Z_k = A^T W_k J_k M_k^T,$$
$$G_2 U_k = V_{k+1}C_k,$$

Description of the Methods

1. Apply SIMBA/SIMBO to $\mathcal{K}$ or $\mathcal{K}_D$
2. Use recurrences to solve reduced system and update approximate solution
3. Use recurrences to bound residual norm

	–SC	–NS
required operation	A-solve	null-space projection of G_1, G_2
process applied to	$\mathcal{K}$	$\mathcal{K}_D$
depends on spectrum of	$S = G_2A^{-1}G_1^T$	$R = H_1^T A H_2$

Table: Comparison of –SC and –NS versions.

	SPMR	SPQMR
monotonic residual	✓	×
short recurrence	✓	✓
bidiagonalization procedure	SIMBA	SIMBO
depends on	singular values of T	eigenvalues of T
mathematically equivalent to	USYMQR on T	QMR on T

Table: Comparison of properties of SPMR vs. SPQMR. The matrix T denotes either the Schur complement (S) or the generalized reduced Hessian (R)

Numerical Experiments

SPMR-SC vs. USYMQR

We compare applying SPMR-SC to (1) with $f = 0$ and USYMQR [4] applied to $Sy = -g$. In exact arithmetic, every iteration would be the same.

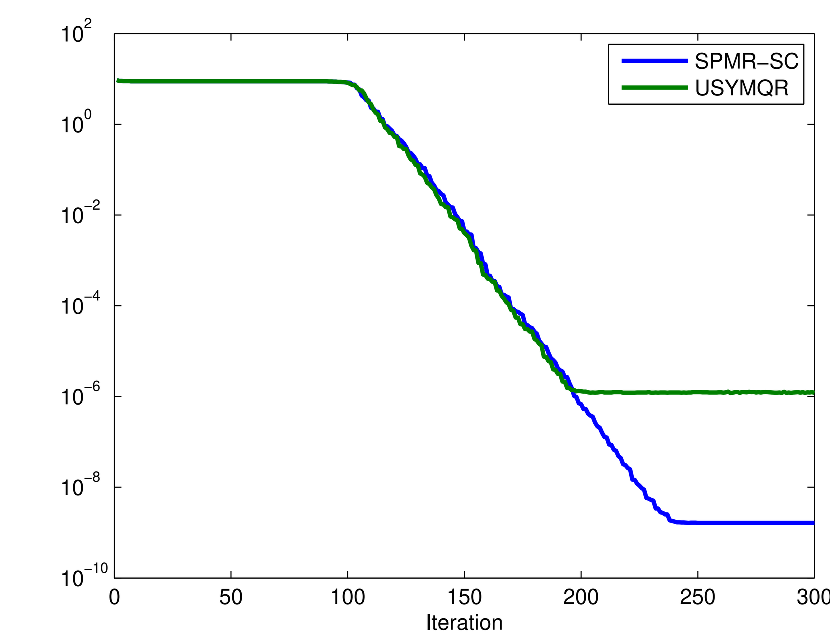


Figure: $\|r_k\|$.

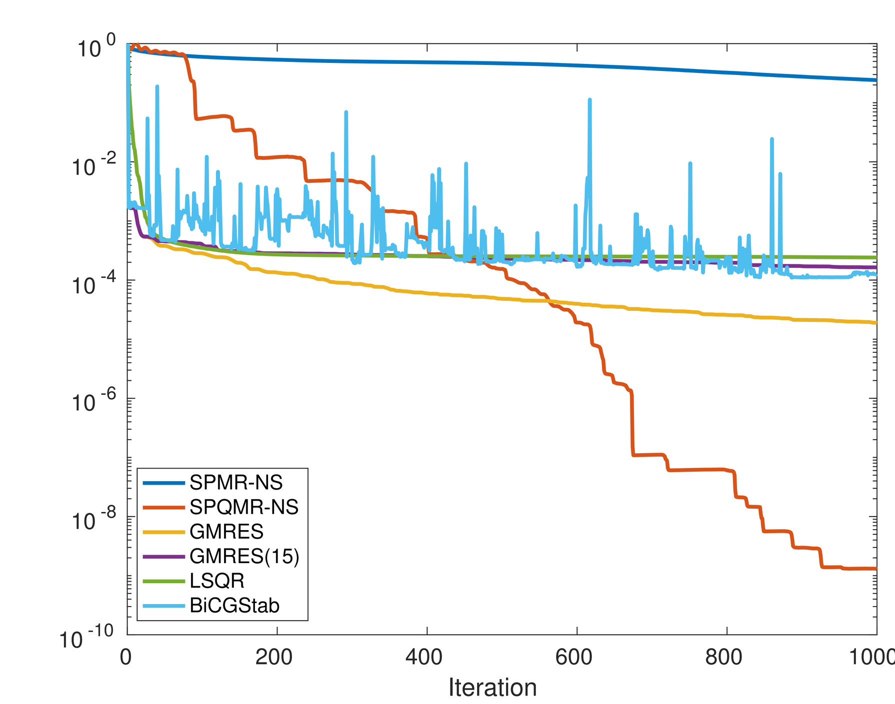
Numerically, SPMR-SC achieves more digits at convergence than USYMQR due to conditioning issues.

Systems from Interior-Point Methods

3×3 block system arising from interior-point methods:

$$\begin{bmatrix}H & -I & J^T\\ -Z & -X & 0\\ J & 0 & 0\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta z\\ -\Delta y\end{bmatrix} = \begin{bmatrix}-c - Hx + J^T y + z\\ b - Jx\\ XZe - \tau e\end{bmatrix}.$$

Apply several iterative methods on ill-conditioned system arising from polygon100 from COPS [1].



References

[1] Alexander S Bondarenko, David M Bortz, and Jorge J Moré. Cops: Large-scale nonlinearly constrained optimization problems. Technical report, Argonne National Lab., IL (US), 2000.

[2] R. Estrin and C. Greif. SPMR: A family of saddle-point minimum residual solvers. *SIAM J. Sci. Comput.*, 2018, accepted for publication.

[3] Roland W. Freund and Noël M. Nachtigal. QMR: a quasi-minimal residual method for non-Hermitian linear systems. *Numerische Mathematik*, 60(1):315–339, 1991.

[4] M. A. Saunders, H. D. Simon, and E. L. Yip. Two conjugate-gradient-type methods for unsymmetric linear equations. *SIAM J. Numer. Anal.*, 25(4):927–940, 1988.